ON THE FOURIER COEFFICIENTS OF HILBERT MODULAR FORMS OF HALF INTEGRAL WEIGHT OVER ALGEBRAIC NUMBER FIELDS (Automorphic forms, automorphic representations and automorphic L-functions over algebraic groups)

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Introduction.

Waldspurger [11] first found that a very interesting relation between Fourier coefficients of modular forms of half integral weight and critical values of twisted L-functions (cf. [2], [3] and [4]). In [8], Shimura succeeded in generalizing such a relation to the case of Hilbert modular forms of half integral weight over totally real number fields. In [5], we derived this in the case of Fourier coefficients of Maass wave forms of half integral weight over an imaginary quadratic field.

The purpose of this paper is to derive a generalization of Shimura's results concerning Fourier coefficients of Hilbert modular forms of half integral weight over total real number fields in the case of Hilbert modular forms over algebraic number fields by following the Shimura's method (cf. [6], [8]). Employing theta functions, we shall construct the Shimura correspondence Ψ_τ from Hilbert forms f of half integral weight over algebraic number fields to Hilbert modular forms $\Psi_\tau(f)$ of integral weight over algebraic number fields. We shall determine explicitly the Fourier coefficients of $\Psi_\tau(f)$ in terms of these of f. Moreover, under some assumptions about f concerning the multiplicity one theorem with respect to Hecke operators, we shall deduce an explicit connection between the square of Fourier coefficients of modular forms f of half integral weight over algebraic number fields and the critical value of the zeta function associated with the image $\Psi_\tau(f)$ of f by the Shimura correspondence Ψ_τ. A possibility of an existence of such a relation was also pointed out by Bump-Friedberg-Hoffstein [1, p.107-p.108] in the case of Maass wave forms of half integral weight over the imaginary quadratic field $\mathbb{Q}(\sqrt{-1})$ from the veiwpoint that the Waldspurger's theorem in this case is equivalent to the assertion that a Rankin-Selberg convolution of two metaplectic forms on $GL(2,\mathbb{C})$ is equal to the Novodvorsky's integral of a metaplectic Eisenstein series on $GSp(4)$ formed with the corresponding non-metaplectic forms. As a consequence of our results, we can solve affirmatively a question of Bump-Friedberg-Hoffstein [1] in the case of Hilbert modular forms of half integral weight over arbitrary algebraic number fields under the assumption that the multiplicity one theorem of Hecke operators is satisfied.

§0 Notation and preliminaries. We denote by $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ the ring of rational integers, the rational number field, the real number field and the complex number field, respectively. For an associative ring R with identity element we denote by $R^\times$ the group of all its invertible elements and by $M_n(R)$ the ring of $n\times n$ matrices with entries in R. Let $GL_n(R')$(resp. $SL_n(R')$) denote the general linear group (resp. special linear group) of degree n over a commutative ring R'. For $x = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(R)$, we put $a = a_x, b = b_x, c = c_x$ and $d = d_x$. Let $\mathbb{H} = \mathbb{R}+\mathbb{R}i+\mathbb{R}j+\mathbb{R}k = \mathbb{C}+\mathbb{C}j$ be the Hamilton quaternion algebra. We denote by $\overline{x} = a - bi - cj - dk$ and $|x| = \sqrt{a^2+b^2+c^2+d^2}$ the conjugate and the absolute value of a quaternion $x = a + bi + cj + dk \in \mathbb{H}$. Throughout this paper, we fix an algebraic number field F of degree d of class number h_F and denote by $a, h, \mathfrak{o}, d_F$ and $\mathfrak{d}$, the set of all archimedean primes, the set of all non archimedean primes, the maximal order of F, the discriminant of F and the different of F relative to $\mathbb{Q}$. Moreover, we denote by s(resp. c) the set of all real (resp. complex) archimedean primes. For an algebraic group $\mathfrak{G}$ defined over F, we define $\mathfrak{G}_v$ for every $v \in a \cup h$ and the adelization $\mathfrak{G}_\mathbb{A}$ of $\mathfrak{G}$ and consider $\mathfrak{G}$ as a subgroup of $\mathfrak{G}_\mathbb{A}$. For an element x of $\mathfrak{G}_\mathbb{A}$ its a-component, s-component, c-component, h-component and v-component are denoted by x_a, x_s, x_c, x_h and x_v. For a fractional ideal $\mathfrak{x}$ in F and $t \in F_\mathbb{A}^\times$ we denote by $N(\mathfrak{x})$ the norm of $\mathfrak{x}$ and by $t\mathfrak{x}$ the fractional ideal in F satisfying $(t\mathfrak{x})_v = t_v\mathfrak{x}_v$ for each $v \in h$. For $v \in h$, we put $N_v = N(\pi_v\mathfrak{o})$ with any prime element π_v of F_v. We consider a continuous character $\psi : F_\mathbb{A}^\times \longrightarrow \mathbb{T} = \{t \in \mathbb{C} | |t| = 1\}$ such that $\psi(F^\times) = 1$. We call ψ a Hecke character of F. Given such a ψ, we denote by ψ^* the ideal character such that

$$\psi^*(t\mathfrak{o}) = \psi(t) \quad \text{if } t \in F_v^\times \text{ and } \psi(\mathfrak{o}_v^\times) = 1 \tag{0-1}$$

and we set $\psi^*(\mathfrak{a}) = 0$ for every fractional ideal $\mathfrak{a}$ that is not prime to the conductor of ψ. For $\psi_v, \psi_a, \psi_s, \psi_c$ and ψ_h, we mean the restriction of ψ on $F_v^\times, F_a^\times, F_s^\times, F_c^\times$ and $F_h^\times$, respectively. For an integral ideal $\mathfrak{z}$ divisible by the conductor $\mathfrak{c}$ of ψ, we put $\psi_\mathfrak{z}(x) = \prod_{v|\mathfrak{z}} \psi_v(x_v)$ for $x = (x_v) \in F_\mathbb{A}^\times$.

§1 Hilbert modular forms of half integral weight over algebraic number fields. We introduce Hilbert Maass forms of half integral weight over an algebraic number field and Hecke operators which act on the space of those. We put

$$H = \{z \in \mathbb{C} | \Im(z) > 0\} \text{ and } H' = \{\mathfrak{z} = z + wj \in \mathbb{H} | z \in \mathbb{C} \text{ and } 0 < w \in \mathbb{R}\}. \tag{1-1}$$

We define an action of $g \in GL_2(\mathbb{C})$ (resp. $GL_2^+(\mathbb{R}) = \{g \in GL_2(\mathbb{R}) | \det g > 0\}$) on H' (resp. H) by

$$\mathfrak{z} \longrightarrow g(\mathfrak{z}) = (a'\mathfrak{z} + b')(c'\mathfrak{z} + d')^{-1} \quad \text{for all } \mathfrak{z} \in H' \tag{1-2}$$

and $g \in GL_2(\mathbb{C})$ with $\frac{1}{\sqrt{\det g}} g = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$ and

$$z \longrightarrow g(z) = (az + b)(cz + d)^{-1} \quad \text{for all } z \in H$$

and $g \in GL_2^+(\mathbb{R})$. For $\mathfrak{z} = z + jw \in H'$(resp. $z \in H$) and $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{C})$ (resp. $GL_2^+(\mathbb{R})$), put

$$\text{(1-3)} \qquad \mu_0(g,\mathfrak{z}) = c\mathfrak{z} + d, m(g,\mathfrak{z}) = |\mu_0(g,\mathfrak{z})|^2 = |cz+d|^2 + |c|^2w^2, w(\mathfrak{z}) = w$$

for all $\mathfrak{z} \in H'$ and $g \in GL_2(\mathbb{C})$ and $j(g,z) = (cz+d)$ for all $z \in H$ and $g \in GL_2^+(\mathbb{R})$. We see that H' has an invariant metric $ds^2(\mathfrak{z}) = (dx^2+dy^2+dw^2)/w^2$ and an invariant measure $dm(\mathfrak{z}) = dxdydw/w^3$ with respect to the action of $GL_2(\mathbb{C})$, where $\mathfrak{z} = x + yi + wj \in H'$. The Laplace-Beltrami operator $L_{\mathfrak{z}}$ is given by

$$\text{(1-4)} \qquad L_{\mathfrak{z}} = w^2(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial w^2}) - w\frac{\partial}{\partial w}.$$

We put

$$\text{(1-5)} \qquad G = SL_2(F) \quad \text{and} \quad \tilde{G} = GL_2(F).$$

For a fractional ideal $\mathfrak{x}$ and $\mathfrak{y}$ of F such that $\mathfrak{x}\mathfrak{y} \subset \mathfrak{o}$, we put

$$\text{(1-6)} \qquad \tilde{D}[\mathfrak{x},\mathfrak{y}] = \tilde{G}_a \prod_{v\in h} \tilde{D}_v[\mathfrak{x},\mathfrak{y}],\ \tilde{D}_v[\mathfrak{x},\mathfrak{y}] = \mathfrak{o}[\mathfrak{x},\mathfrak{y}]^\times,\ D[\mathfrak{x},\mathfrak{y}] = G_A \cap \tilde{D}[\mathfrak{x},\mathfrak{y}],$$

$$D_v[\mathfrak{x},\mathfrak{y}] = G_v \cap \tilde{D}_v[\mathfrak{x},\mathfrak{y}],\ \tilde{\Gamma}[\mathfrak{x},\mathfrak{y}] = \tilde{G} \cap \tilde{D}[\mathfrak{x},\mathfrak{y}] \text{ and } \Gamma[\mathfrak{x},\mathfrak{y}] = G \cap D[\mathfrak{x},\mathfrak{y}],$$

where

$$\text{(1-7)} \qquad \mathfrak{o}[\mathfrak{x},\mathfrak{y}] = \left\{ x = \begin{pmatrix} a_x & b_x \\ c_x & d_x \end{pmatrix} \in M_2(F) \middle| a_x \in \mathfrak{o}, b_x \in \mathfrak{x}, c_x \in \mathfrak{y} \text{ and } d_x \in \mathfrak{o} \right\}.$$

Let r_1(resp. r_2) be the cardinal number of c (resp. s). For $i(1 \leqq i \leqq r_1)$ (resp. $i'(1 \leqq i' \leqq r_2)$), we choose a $v = v_i \in s$(resp. $v' = v_{r_1+i'} \in c$) such that $v_i \neq v_j (i \neq j)$(resp. $v_{r_1+i'} \neq v_{r_1+j'} (i' \neq j')$). We put

$$\text{(1-8)} \quad \tilde{G}_{a^+} = \{g = (g_1, \cdots, g_{r_1}, g_{r_1+1}, \cdots, g_{r_1+r_2}) \in \tilde{G}_a |\det(g_i) > 0 (1 \leqq i \leqq r_1)\}$$

$$\tilde{G}_{\mathbb{A}^+} = \tilde{G}_{a^+}\tilde{G}_h, \tilde{G}_+ = \tilde{G}_{\mathbb{A}^+} \cap \tilde{G} \text{ and } D = H^{r_1} \times (H')^{r_2}.$$

We define an action $\tilde{G}_{a^+}$ on D by

$$\text{(1-9)} \qquad \mathfrak{z} = (z_1, \cdots, z_{r_1}, \mathfrak{z}_{r_1+1}, \cdots, \mathfrak{z}_{r_1+r_2}) \in D \to g(\mathfrak{z}) =$$

$$(g_1(z_1), \cdots, g_{r_1}(z_{r_1}), g_{r_1+1}(\mathfrak{z}_{r_1+1}), \cdots, g_{r_1+r_2}(\mathfrak{z}_{r_1+r_2}))$$

for each $g = (g_1, \cdots, g_{r_1}, g_{r_1+1}, \cdots, g_{r_1+r_2}) \in \tilde{G}_{a^+}$. We denote by $M_p(F_{\mathbb{A}})$ the metaplectic group of Weil [12] with respect to the alternating form $(x,y) \longrightarrow x\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} {}^t y$ on F^2. There exists an exact sequence

$$\text{(1-10)} \qquad 1 \longrightarrow \mathbb{T} \longrightarrow M_p(F_{\mathbb{A}}) \longrightarrow G_{\mathbb{A}} \longrightarrow 1$$

and a natural lift $r : G \longrightarrow M_p(F_{\mathbb{A}})$ by which we may view G as a subgroup of $M_p(F_{\mathbb{A}})$. We denote by pr the projection map of $M_p(F_{\mathbb{A}})$ to $G_{\mathbb{A}}$. For $\tau \in \mathrm{pr}^{-1}(P_A C'')$ and $\mathfrak{z} \in D = H^{r_1} \times (H')^{r_2}$, we denote by $h(\tau, \mathfrak{z})$ the quasi factor of automorphy of weight 1/2 defined in Shimura [9, p.1021], where

$$P = \left\{ \alpha = \begin{pmatrix} a_\alpha & b_\alpha \\ c_\alpha & d_\alpha \end{pmatrix} \in G \middle| c_\alpha = 0 \right\}, \quad C' = D[2\mathfrak{d}^{-1}, 2\mathfrak{d}],$$

$$C'' = C' \cup C'\epsilon, \epsilon \in G_{\mathbb{A}}, \epsilon_a = 1 \text{ and } \epsilon_v = \begin{pmatrix} 0 & -\delta_v^{-1} \\ \delta_v & 0 \end{pmatrix} (v \in h)$$

with an arbitrary fixed element $\delta \in F_h^\times$ such that $\mathfrak{d} = \delta\mathfrak{o}$. We refer to [9] and [12] for details. For $\tau \in \mathrm{pr}^{-1}(P_{\mathbb{A}} C''), m \in \mathbb{Z}^{r_1}$ and $\mathbb{C}$-valued function f on D, we define a function $f||_{m+(1/2)u_{r_1}} \tau$ on D by

(1-11) $\qquad (f||_{m+(1/2)u_{r_1}} \tau)(\mathfrak{z}) = J_m(\tau, \mathfrak{z})^{-1} f(\tau(\mathfrak{z})) \quad$ for all $\mathfrak{z} = (z_1, \cdots, z_{r_1},$

$\mathfrak{z}_{r_1+1}, \cdots, \mathfrak{z}_{r_1+r_2}) \in D$, where $u_{r_1} = (1, \cdots, 1) \in \mathbb{Z}^{r_1}$ and

$$J_m(\tau, \mathfrak{z}) = h(\tau, \mathfrak{z})^{-3} \prod_{i=1}^{r_1} j(c_{\tau_i} z_i + d_{\tau_i})^{m_i+2} \prod_{i=1}^{r_2} m(\tau_{r_1+i}, \mathfrak{z}_{r_1+i})^3.$$

Here we write τ for $\mathrm{pr}(\tau)$. Let ψ be a Hecke character of the conductor $\mathfrak{f}$ and let $\mathfrak{b}, \mathfrak{b}'$ be two integral ideals of F such that $\mathfrak{f}$ divides $4\mathfrak{b}\mathfrak{b}'$. For $\omega = (\omega_{r_1+1}, \cdots, \omega_{r_1+r_2}) \in \mathbb{C}^{r_2}$, we consider $\mathbb{C}$-valued real analytic function f on D satisfying the following condition

(1-12) $\quad (i) f||_{m+(1/2)u_{r_1}} \gamma(\mathfrak{z}) = \psi_{\mathfrak{f}}(a_\gamma) f(\mathfrak{z})$ for every $\gamma \in \Gamma[2\mathfrak{b}\mathfrak{d}^{-1}, 2\mathfrak{b}'\mathfrak{d}]$ and $\mathfrak{z} \in D$,

(ii) $L_{\mathfrak{z}_{r_1+i}}(w_{r_1+i}^{3/2} f(\mathfrak{z})) = \omega_{r_1+i} w_{r_1+i}^{3/2} f(\mathfrak{z})$ $(1 \leqq i \leqq r_2)$ and $f(\mathfrak{z})$ is a holomorphic function with respect to $z_1, \cdots, z_{r_1}$,

(iii) f vanishes at each cusp of $\Gamma[2\mathfrak{b}\mathfrak{d}^{-1}, 2\mathfrak{b}'\mathfrak{d}]$,

where $\mathfrak{z} = (z_1, \cdots, z_{r_1}, \mathfrak{z}_{r_1+1}, \cdots, \mathfrak{z}_{r_1+r_2}) \in D$ and $\mathfrak{z}_{r_1+i} = z_{r_1+i} + jw_{r_1+i} (1 \leqq i \leqq r_2)$. See Zhao [10] and Shimura [6] for the condition (iii). We denote by $\mathcal{S}_{m+(1/2)u_{r_1}, \omega}(\mathfrak{b}, \mathfrak{b}' : \psi)$ the set of all such functions f. We call such a f a modular cusp form of half integral weight $m + (1/2)u_{r_1}$. For two cusp forms f and g of weight $m + (1/2)u_{r_1}$ with respect to a congruence subgroup Γ of G, we determine their inner product $\langle f, g \rangle$ by

(1-13) $$\langle f, g \rangle = \mathrm{vol}(\Gamma \backslash D)^{-1} \int_{\Gamma \backslash D} \overline{f(\mathfrak{z})} g(\mathfrak{z}) \Im(\mathfrak{z})^{m+(1/2)u_{r_1}} w^3 d\mathfrak{z},$$

where $\mathfrak{z} = (z_1, \cdots, z_{r_1}, \mathfrak{z}_{r_1+1}, \cdots, \mathfrak{z}_{r_1+r_2}) \in D, \Im(\mathfrak{z})^{m+(1/2)u_{r_1}} = \prod_{i=1}^{r_1} (\Im z_i)^{m_i+1/2}$, $w = \prod_{i=1}^{r_2} w_{r_1+i}$ and $\mathfrak{z}_{r_1+i} = z_{r_1+i} + jw_{r_1+i}$. Given $f \in \mathcal{S}_{m+(1/2)u_{r_1}, \omega}(\mathfrak{b}, \mathfrak{b}'; \psi)$, we define a function $f_{\mathbb{A}}$ on $M_p(F_{\mathbb{A}})$ by

(1-14) $\qquad f_{\mathbb{A}}(\alpha x) = (f||_{m+(1/2)u_{r_1}} x)(j') \quad$ for every $\alpha \in G$ and $x \in M_p(F_{\mathbb{A}})$

such that $\mathrm{pr}(x) \in B$, where $j' = (\overbrace{i, \cdots, i}^{r_1}, \overbrace{j, \cdots, j}^{r_2}) \in D$ and B is an open subgroup of C'' satisfying $f||_{m+(1/2)u_{r_1}}\gamma = f$ for every $\gamma \in B \cap G$. We have

(1-15) $$f_{\mathbb{A}}(\alpha x w) = \psi_{\mathfrak{f}}(a_w)^{-1} J_m(w, \mathfrak{z})^{-1} f_{\mathbb{A}}(x) \quad \text{for every } \alpha \in G$$

and $w \in D[2\mathfrak{b}\mathfrak{d}^{-1}, 2\mathfrak{b}'\mathfrak{d}]$ such that $w(j') = j'$. We define a map $e : \mathbb{C} \longrightarrow \mathbb{C}$ and characters $e_{\mathbb{A}}$ and e_v of $F_{\mathbb{A}}$ and F_v by

$$e[z] = \exp(2\pi i z)(z \in \mathbb{C}), \;\; e_{\mathbb{A}}(x) = \prod_{v \in a \cup h} e_v(x) \text{ for } x = (x_v) \in F_{\mathbb{A}},$$

$e_v(x) = e[x_v]$ for $v \in s$, $e_v(x) = e[x_v + \overline{x_v}]$ for $v \in c$ and $e_v(x_v) = e[-y]$ for $v \in h$, where $y \in \cap_{q \neq p}(\mathbb{Z}_q \cap \mathbb{Q})$, $\;y - \mathrm{Tr}_{F_v/\mathbb{Q}_p}(x_v) \in \mathbb{Z}_p$, $\;v|p$. We put

$$e_a(x) = e_{\mathbb{A}}(x_a), e_s(x) = e_{\mathbb{A}}(x_s), e_c(x) = e_{\mathbb{A}}(x_c), e_h(x) = e_{\mathbb{A}}(x_h) \text{ and}$$

$$\tilde{K_\lambda}(v) = \prod_{i=1}^{r_2} (4\pi |v_{r_1+i}|)^{-1/2} K_\lambda(4\pi |v_{r_1+i}|) \;\; (v = (v_{r_1+1}, \cdots, v_{r_1+r_2}) \in F_c^\times)$$

with $\lambda = (\lambda_1, \cdots, \lambda_{r_2}) \in \mathbb{C}^{r_2}$, where $K_\lambda(v) = 2^{-1}\int_0^\infty \exp(-2^{-1}v(t+t^{-1}))t^{-1-\lambda}dt$ ($v \in \mathbb{R}^+ = \{v \in \mathbb{R} | v > 0\}$) with $\lambda \in \mathbb{C}$. Then we have the following lemma (cf. [6, Prop. 3.1]).

Lemma 1.1. *Suppose $f \in S_{m+(1/2)u_{r_1},\omega}(\mathfrak{b}, \mathfrak{b}'; \psi)$. Then there is a complex number $\mu(\xi, \mathfrak{m}; f, \psi)$ determined for $\xi \in F$ and a fractional ideal $\mathfrak{m}$ in F such that*

(1-16)
$$\psi_{\mathfrak{f}}(t) f_{\mathbb{A}}(r_p \begin{pmatrix} t & s \\ 0 & t^{-1} \end{pmatrix}) = |t|_{\mathbb{A}}^{1/2} t_s^m |t_c|' \sum_{\xi \in F^\times} \mu(\xi, t\mathfrak{o}; f, \psi) e_s(it^2\xi/2) \tilde{K_\nu}(\xi |t_c|^2/2) e_{\mathbb{A}}(\xi t s/2),$$

where $|t_c|' = \prod_{i=1}^{r_2} |t_{r_1+i}|^2 (t_c = (t_{r_1+1}, \cdots, t_{r_1+r_2}))$, $|t|_{\mathbb{A}} = \prod_{v \in a \cup h} |t|_v, |t|_v = |t_v|_v$ is the normalized valuation $|\;|_v$ at v with $t = (t_v) \in F_{\mathbb{A}}^\times$ and $t_s^m = \prod_{i=1}^{r_1} t_i^{m_i} (t_s = (t_i) \in F_s^\times)$. Moreover, $\mu(\xi, \mathfrak{m}; f, \psi)$ holds the following properties:

(1-17) $$\mu(\xi, \mathfrak{m}; f, \psi) \neq 0 \quad \textit{only if} \quad \xi \in \mathfrak{b}^{-1}\mathfrak{m}^{-2} \;\; \textit{and} \;\; \xi \neq 0 \;\textit{and}$$

$$\mu(\xi b^2, \mathfrak{m}; f, \psi) = b_s^m |b_c|^2 \psi_a(b) \mu(\xi, b\mathfrak{m}; f, \psi) \quad \textit{for every } b \in F^\times,$$

where $b_s^m = \prod_{i=1}^{r_1} (b^{(i)})^{m_i}, |b_c| = \prod_{i=1}^{r_2} |b^{(r_1+i)}|$ and $b_a = (b^{(1)}, \cdots, b^{(r_1)}, b^{(r_1+1)}, b^{(r_1+r_2)})$. Furthermore, $\beta \in G \cap diag[r, r^{-1}]D[2\mathfrak{b}\mathfrak{d}^{-1}, 2\mathfrak{b}'\mathfrak{d}]$ with $r \in F_{\mathbb{A}}^\times$, then

(1-18)
$$\psi_a(d_\beta)\psi^*(d_\beta \mathfrak{a}_\beta^{-1}) f(\beta^{-1}(\mathfrak{z})) N(\mathfrak{a}_\beta)^{1/2} = J_m(\beta, \beta^{-1}(\mathfrak{z}))^{-1} \sum_{\xi \in F^\times} \mu(\xi, \mathfrak{a}_\beta^{-1}; f, \psi) e_s(\xi z_1/2) \tilde{K_\nu}(\xi_c w/2) e_c(\xi z_2/2),$$

where $\mathfrak{a}_\beta = r^{-1}\mathfrak{o}, \mathfrak{z} = (z_1, \cdots, z_{r_1}, \mathfrak{z}_{r_1+1}, \cdots, \mathfrak{z}_{r_1+r_2}) \in D$, $\xi z_1 = (\xi^{(1)}z_1, \cdots, \xi^{(r_1)}z_{r_1})$, $\mathfrak{z}_{r_1+i} = z_{r_1+i} + jw_{r_1+i}$, $\xi_c w = (\xi^{(r_1+1)}w_{r_1+1}, \cdots, \xi^{(r_1+r_2)}w_{r_1+r_2})$ *and* $\xi_c z_2 = (\xi^{(r_1+1)}z_{r_1+1}, \cdots, \xi^{(r_1+r_2)}z_{r_1+r_2})$.

We simply write $\mu_f(\xi, \mathfrak{m})$ for $\mu(\xi, \mathfrak{m}; f, \psi)$. We denote by $\{\mathbb{T}_v\}_{v \in h}$ the Hecke operators on $\mathcal{S}_{m+(1/2)u_{r_1}, \omega}(\mathfrak{b}, \mathfrak{b}'; \psi)$ which are defined by the same manner as that in [8, p.510]. Let Ψ' be a Hecke character whose conductor divides an integral ideal $\mathfrak{i}$. Moreover, we assume that

$$\Psi'(x) = \prod_{i=1}^{r_1} (\mathrm{sgn}(x_i))^{n_i} |x_i|^{2\sqrt{-1}\lambda_i} \prod_{i=1}^{r_2} |x_{r_1+i}|^{4\sqrt{-1}\mu_{r_1+i}} \tag{1-19}$$

$(x = (x_1, \cdots, x_{r_1}, x_{r_1+1}, \cdots, x_{r_1+r_2}) \in F_a^\times)$ such that $\lambda_i, \mu_{r_1+j} \in \mathbb{R} (1 \leqq i \leqq r_1, 1 \leqq j \leqq r_2), n = (n_1, \cdots, n_{r_2}) \in \mathbb{Z}^{r_1}$ and $\sum_{i=1}^{r_1} \lambda_i + \sum_{j=1}^{r_2} \mu_{r_1+j} = 0$. We put $\tilde{D}_\mathfrak{i} = \tilde{D}[\mathfrak{d}^{-1}, \mathfrak{i}\mathfrak{d}]$. We consider a $\mathbb{C}$-valued function $\mathfrak{g}$ on $\tilde{G}_\mathbb{A}$ satisfying the following conditions

(1-20) (*i*) $\mathfrak{g}(sx) = \Psi'(s)\mathfrak{g}(x)$ for every $s \in F_\mathbb{A}^\times$ and $x \in \tilde{G}_\mathbb{A}$,

(*ii*) $\mathfrak{g}(\alpha x w) = \Psi'((d_w)_\mathfrak{i})\mathfrak{g}(x)$ for every $\alpha \in \tilde{G}, x \in \tilde{G}_\mathbb{A}$ and $w \in \tilde{D}_\mathfrak{i}$,

where $w_a = 1$ and $d_\mathfrak{i} = (d_v)_{v|\mathfrak{i}}$ for $d \in F_\mathbb{A}$.

(*iii*) There exists a function g_λ on D such that

$$\mathfrak{g}(t_\lambda^{-1} x_\lambda y) = \det(r)_s^{i\lambda} |\det(r)_c|^{2i\mu} \prod_{i=1}^{r_i} j(y_i, \sqrt{-1})^{n_i} g_\lambda(y(j'))$$

for every $y \in \tilde{G}_a$, where $\tilde{G}_\mathbb{A} = \bigsqcup_{i=1}^{\kappa} \tilde{G} x_\lambda \tilde{D}_\mathfrak{z}, x_\lambda = \mathrm{diag}[1, t_\lambda] (t_\lambda \in F_h^\times)$. Moreover g_λ satisfies the following conditions

$$\begin{aligned} (iv) \quad g_\lambda \|_n \gamma(\mathfrak{z}) &= (\det(\gamma)^{-1/2})^{-n} (c_\gamma z + d_\gamma)^{-n} g_\lambda(\gamma(\mathfrak{z})) \\ &= \Psi'_\mathfrak{i}(a_\gamma)(\det(r)_s)^{i\lambda} (\det(r)_c)^{2i\mu} g_\lambda(\mathfrak{z}) \end{aligned}$$

for every $\gamma \in \tilde{\Gamma}[(t_\lambda \mathfrak{d})^{-1}, t_\lambda \mathfrak{i}\mathfrak{d}]$ and $\mathfrak{z} \in D$

$$(v) \quad g_\lambda(\mathfrak{z}) = \sum_{0 \neq \xi \in F} c_\lambda(\xi) e_s(\xi z) w K_{\nu'}(4\pi|\xi|w) e_c(\xi z'),$$

where $\nu' = (\nu'_{r_1+1}, \cdots, \nu'_{r_1+r_2}), \omega' = (\omega'_{r_1+1}, \cdots, \omega'_{r_1+r_2}), (\nu')^2_{r_1+i} = \omega'_{r_1+i} + 1$ $(1 \leqq i \leqq r_2)$, $\mathfrak{z} = (z_1, \cdots, z_{r_1}, z_{r_1+1} + jw_{r_1+1}, \cdots, z_{r_1+r_2} + jw_{r_1+r_2})$, $e_s(\xi z) = \prod_{i=1}^{r_1} e[\xi^{(i)} z_i], w = \prod_{i=1}^{r_2} w_{r_1+i}, K_{\nu'}(4\pi|\xi|w) = \prod_{i=1}^{r_2} K_{\nu'_{r_1+i}}(4\pi|\xi^{(r_1+i)}|w_{r_1+i})$ and $e_c(\xi z') = \prod_{i=1}^{r_2} e[2\Re(\xi^{(r_1+i)} z_{r_1+i})]$. From the above conditions, $\mathfrak{g}$ is determined by

$(g_1, \cdots, g_\kappa)$. We denote by $\mathcal{S}_{n,\omega'}(\mathfrak{i}, \Psi')$ the set of all such $\mathfrak{g}$. For a fractional ideal $\mathfrak{m}$, determine $c(\mathfrak{m}, \mathfrak{g})$ by

$$c(\mathfrak{m}, \mathfrak{g}) = c_\lambda(\xi)\xi^{-(n/2)-i\lambda}|\xi|^{-1-2\mu i} \quad \text{if } \mathfrak{m} = \xi t_\lambda^{-1}\mathfrak{o}. \tag{1-21}$$

Then we have

(1-22)

$$\mathfrak{g}(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}) = \sum_{0 \ll \xi \in F^\times} c(\xi y\mathfrak{o}, \mathfrak{g})(\xi y)^{n/2+i\lambda}|\xi y_\infty|^{2\mu i} e_s(i\xi y)\tilde{\tilde{K}}_{\nu'}((\xi y_\infty)_c)e_{\mathbb{A}}(\xi x)$$

with $\tilde{\tilde{K}}_{\nu'}(v) = |v|K_{\nu'}(4\pi|v|)(v \in \mathbb{C}^\times)$. For each integral ideal $\mathfrak{n}$ in F we can define a $\mathbb{C}$-linear endomorphism $\mathfrak{T}(\mathfrak{n})$ of $\mathcal{S}_{n,\omega'}(\mathfrak{i}, \Psi')$ such that

$$c(\mathfrak{m}, \mathfrak{g}|\mathfrak{T}(\mathfrak{n})) = \sum_{\mathfrak{a} \supset \mathfrak{M}+\mathfrak{n}} \Psi'_*(\mathfrak{a})N(\mathfrak{a}^{-1}\mathfrak{n})c(\mathfrak{a}^{-2}\mathfrak{m}\mathfrak{n}, \mathfrak{g}), \tag{1-23}$$

where $\Psi'_*(\mathfrak{a})$ denotes $\Psi'^*(\mathfrak{a})$ or 0 according as $\mathfrak{a}$ is prime to $\mathfrak{i}$ or not. Let $\mathfrak{g}$ be a common eigenform of $\mathfrak{T}(\mathfrak{n})$ for all integral ideals $\mathfrak{n}$; Put $\mathfrak{g}|\mathfrak{T}(\mathfrak{n}) = \chi'(\mathfrak{n})\mathfrak{g}$ and $\chi'(v) = \chi'(\pi_v\mathfrak{o})$ for every $v \in h$. Then we call χ' a system of eigenvalues. We call such an eigenform $\mathfrak{g}$ normalized if $c(\mathfrak{o}, \mathfrak{g}) = 1$. For a Hecke character ρ of F and an integral ideal $\mathfrak{r}$, we put

$$D(s, \chi', \rho) = \sum_{\mathfrak{m}} \rho^*(\mathfrak{m})\chi'(\mathfrak{m})N(\mathfrak{m})^{-s-1}, \quad D(s, \chi') = \sum_{\mathfrak{m}} \chi'(\mathfrak{m})N(\mathfrak{m})^{-s-1}, \tag{1-24}$$

where the summation $\sum_{\mathfrak{m}}$ is taken over all integral ideals $\mathfrak{m}$. We can also define an inner product $\langle \mathfrak{g}, \mathfrak{g}' \rangle$ for every $\mathfrak{g}, \mathfrak{g}' \in \mathcal{S}_{n,\omega'}(\mathfrak{i}, \Psi')$.

§2 Shimura correspondence of modular forms of half integral weight. The purpose of this section is to introduce the Shimura correspondence Ψ_τ of Hilbert modular forms f of half integral weight over an algebraic number field to those $\Psi_\tau(f)$ of integral weight and to determine the explicit Fourier coefficients of $\Psi_\tau(f)$ in terms of those of f. Let F be an algebraic number field with r_1 real archimedean primes and r_2 complex archimedean primes. We consider the imbedding F into $\mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$ defined by

$$\alpha \in F \to (\alpha^{(1)}, \cdots, \alpha^{(r_1)}, \alpha^{(r_1+1)}, \cdots, \alpha^{(r_1+r_2)}) \in \mathbb{R}^{r_1} \times \mathbb{C}^{r_2}.$$

For $\mathfrak{z} = (z_1, \cdots, z_{r_1}, \mathfrak{z}_{r_1+1}, \cdots, \mathfrak{z}_{r_1+r_2})$ and $\mathfrak{w} = (z'_1, \cdots, z'_{r_1}, \mathfrak{z}'_{r_1+1}, \cdots, \mathfrak{z}'_{r_1+r_2}) \in D$, $\xi \in V = \{\xi \in M_2(F)|\ \mathrm{tr}(\xi) = 0\}$ and $m = (m_1, \cdots, m_{r_1}) \in \mathbb{Z}^{r_1}(m_i \geqq 0)$, put

$$\begin{aligned} \Psi(\xi, \mathfrak{z}, \mathfrak{w}) = e[&\sum_{i=1}^{r_1}\{2^{-1}\det(\xi^{(i)})z_i + 4^{-1}\sqrt{-1}\Im(z_i) \\ &\times |[\xi^{(i)}, z'_i]/\eta(z'_i)|^2\} + \sum_{i=1}^{r_2}\{\Re(\det(\xi^{(r_1+i)})z_i) + \sqrt{-1}w_{r_1+i} \\ &\times (\Re(\det(\xi^{(r_1+i)})) + 2^{-1}|[\xi^{(r_1+i)}, \mathfrak{z}'_{r_1+i}]/\eta(\mathfrak{z}'_{r_1+i})|^2)\}] \end{aligned} \tag{2-1}$$

$$\text{and} \quad \Psi(\xi, \mathfrak{w}) = \prod_{i=1}^{r_1} [\xi^{(i)}, \overline{z_i'}]^{m_i},$$

where $\mathfrak{z}_{r_1+i} = z_{r_1+i} + jw_{r_1+i}, \mathfrak{z}'_{r_1+i} = z'_{r_1+i} + jw'_{r_1+i}, \eta(z_i') = \Im(z_i'), \eta(\mathfrak{z}'_{r_1+i}) = w'_{r_1+i}$,

$$[\xi, z] = [\xi, z, z] \text{ and } [\xi; \mathfrak{w}, \mathfrak{w}'] = (-1 \ \ \mathfrak{w})\xi \begin{pmatrix} \mathfrak{w}' \\ 1 \end{pmatrix} (\in \mathbb{H}) \quad \text{for every } \mathfrak{w}, \mathfrak{w}' \in \mathbb{H}.$$

Define a theta function $\Theta(\mathfrak{z}, \mathfrak{w}, \lambda)$ on $D \times D$ by

$$\Theta(\mathfrak{z}, \mathfrak{w}, \lambda) = \prod_{i=1}^{r_1} \Im(z_i)^{1/2} \Im(z_i')^{-2m_i} \sum_{\xi \in V} \lambda(\xi_h) \Psi(\xi, \mathfrak{w}) \Psi(\xi, \mathfrak{z}, \mathfrak{w}) \tag{2-2}$$

for every $\mathfrak{z}, \mathfrak{w} \in D$ and $\lambda \in \mathcal{S}(V_h)$, where $\mathcal{S}(V_h)$ means the Schwartz-Bruhat space of V_h. Let $\mathfrak{b}, \mathfrak{b}'$ be integral ideals and let ψ be a Hecke character of F whose conductor divides $4\mathfrak{b}\mathfrak{b}'$. Put $u_{r_1} = (1, \cdots, 1) \in \mathbb{Z}^{r_1}$, $u_{r_2} = (1, \cdots, 1) \in \mathbb{Z}^{r_2}, m = (m_1, \cdots, m_{r_1}) \in \mathbb{Z}^{r_1} (m_i \geqq 0)$ and $\omega = (\omega_{r_1+1}, \cdots, \omega_{r^1+r^2}) \in \mathbb{C}^{r_2}$. Take a $f \in \mathcal{S}_{m+(1/2)u_{r_1}, \omega}(\mathfrak{b}, \mathfrak{b}' : \psi)$. Let τ be an element of $F^\times$ such that $\tau \gg 0, \tau\mathfrak{b} = \mathfrak{q}^2\mathfrak{r}$ with a fractional ideal $\mathfrak{q}$ and a square free integral ideal $\mathfrak{r}$. We put $\mathfrak{c} = 4\mathfrak{b}\mathfrak{b}', \mathfrak{e} = 2^{-1}\mathfrak{c}\mathfrak{d}$ and $\varphi = \psi\epsilon_\tau$ with the Hecke character ϵ_τ associated with the quadratic extension $F(\sqrt{\tau})/F$. We denote by $\mathfrak{h}$ the conductor of φ. We put

$$\tilde{G}_{\mathbb{A}} = \bigsqcup_{\lambda=1}^{\kappa} \tilde{G} x_\lambda \tilde{D}_{\mathfrak{z}}, x_\lambda = \operatorname{diag}[1, t_\lambda], t_\lambda \in F_h^\times \text{ and } \mathfrak{e}_\lambda = 2^{-1} t_\lambda \mathfrak{c}\mathfrak{d} \ \ (\lambda = 1, \cdots, \kappa),$$

where $\tilde{D}_{\mathfrak{i}} = \tilde{D}[\mathfrak{d}^{-1}, \mathfrak{i}\mathfrak{d}]$. We define an element $\eta \in \mathcal{S}(V_h)$ as follows:

$$\eta(x) = \begin{cases} \sum_t \varphi_a(t) \varphi^*((2t\mathfrak{r})) e_a(-b_x t) & \text{if } x = \begin{pmatrix} a_x & b_x \\ c_x & -a_x \end{pmatrix} \in \mathfrak{o}[\mathfrak{e}^{-1}, \mathfrak{e}], \\ 0 & \text{otherwise,} \end{cases} \tag{2-3}$$

where t runs over all elements of $(2\mathfrak{r})^{-1}/2^{-1}\mathfrak{c}$ satisfying the conditions $2t\mathfrak{r} + \mathfrak{r}\mathfrak{c} = \mathfrak{o}$. For $\xi \in \mathcal{S}(V_h)$, put

$$\xi_\lambda(y) = \varphi(t_\lambda)^{-1} \xi(x_\lambda^{-1} y x_\lambda) \ \ (\lambda = 1, \cdots, \kappa).$$

By virtue of Shimura [9, Prop.5.1], we may derive that
(2-4)

$$\Theta(\gamma(\mathfrak{z}), \mathfrak{w}, \eta_\lambda) = \overline{J_m(\gamma, \mathfrak{z})} \varphi_{\mathfrak{h}}(a_\gamma)^{-1} \Theta(\mathfrak{z}, \mathfrak{w}, \eta_\lambda) \quad \text{for every } \gamma \in \Gamma[2\mathfrak{d}^{-1}, 2^{-1}\mathfrak{r}\mathfrak{c}\mathfrak{d}].$$

Define a function $g_{\tau,\lambda}(\mathfrak{w}) = \Psi_{\tau,\lambda}(f)(\mathfrak{w})$ on D by

$$C g_{\tau,\lambda}(\mathfrak{w}) = \int_{\Gamma_{\mathfrak{r}\mathfrak{c}} \backslash D} h(\mathfrak{z}) \Theta(\mathfrak{z}, \mathfrak{w}; \eta_\lambda) \Im(z)^{m+(1/2)u_{r_1}} w^3 d\mathfrak{z} \tag{2-5}$$

for every $\mathfrak{w} \in D$, where $C = i^{\{m\}}2^{1+r_1-r_2+\{m\}}(1/\sqrt{2\pi})^{r_2}\varphi_a(1/2)N(\mathfrak{rc})$, $\Gamma_{\mathfrak{rc}} = \Gamma[2\mathfrak{d}^{-1}, 2^{-1}\mathfrak{rc}\mathfrak{d}]$, $\mathfrak{z} = (z_1, \cdots, z_{r_1}, \mathfrak{z}_{r_1+1}, \cdots, \mathfrak{z}_{r_1+r_2})$, $i^{\{m\}} = i^{\sum_{i=1}^{r_1} m_i}$, $2^{1+r_1-r_2+\{m\}} = \prod_{i=1}^{r_1} 2^{1+r_1-r_2+m_i}$, $\Im(z)^{m+(1/2)u_{r_1}} = \prod_{i=1}^{r_1} \Im(z_i)^{m_i+1/2}$, $w^3 = \prod_{i=1}^{r_2} w_{r_1+i}^3$ and $\mathfrak{z}_{r_1+i} = z_{r_1+i} + jw_{r_1+i} (1 \leqq i \leqq r_2)$. By the transformation formula (2-4), this integral is meaningful. For the convergence of it we refer to Shimura [6, Prop. 7.1]. By Shimura [6, Prop. 7.1], $g_{T,\lambda}(\mathfrak{w})$ is holomorphic with respect to $z'_1, \cdots, z'_{r_1} \in H$. Combining a very long tedious computation with the self-adjointness of the Laplace Beltrami operators $L_{\mathfrak{z}_{r_1+i}} (1 \leqq i \leqq r_2)$, we confirm that

(2-6) $$L_{\mathfrak{z}'_{r_1+i}} g_{T,\lambda}(\mathfrak{w}) = (4\omega_{r_1+i} + 3) g_{T,\lambda}(\mathfrak{w}) \quad (1 \leqq i \leqq r_2).$$

Next we shall determine explicitly Fourier coefficients of $g_{T,\lambda}(\mathfrak{w})$ in terms of those of f. To execute this, we need to represent $\Theta(\mathfrak{z}, \mathfrak{w}; \eta_\lambda)$ as a Poincarè series-type sum. For $l = (l_1, \cdots, l_{r_1}) \in \mathbb{Z}^{r_1}$ and $u = (u_{r_1+1}, \cdots, u_{r_1+r_2}) \in \mathbb{C}^{r_2}$, we define a theta function $\tilde{\vartheta}_l(\mathfrak{z}, u)$ by

(2-7)
$$\tilde{\vartheta}_l(\mathfrak{z}, u) = N(\mathfrak{a}_\beta)^{1/2} y^{-l/2} \times \sum_{\xi \in \mathfrak{a}_\beta} H_l(\sqrt{4\pi y}\xi) e_c(-\xi u) e_c(\xi^2 z/2) \exp(-2\pi w|\xi|^2) e_s(\xi^2 z/2),$$

where

$$\mathfrak{z} = (z_1, \cdots, z_{r_1}, \mathfrak{z}_{r_1+1}, \cdots, \mathfrak{z}_{r_1+r_2}) \ (\mathfrak{z}_{r_1+i} = z_{r_1+i} + jw_{r_1+i}),$$

$$y^{-l/2} = \prod_{i=1}^{r_1} \Im(z_i)^{-l_i/2}, e_s(\xi^2 z/2) = \prod_{i=1}^{r_1} e_s((\xi^{(i)})^2 z_i/2)$$

$$e_c(-\xi u) = \prod_{i=1}^{r_2} e[-\xi^{(r_1+i)} u_{r_1+i}], e_c(\xi^2 z/2) = \prod_{i=1}^{r_2} e[(\xi^{(r_1+i)})^2 z_{r_1+i}/2],$$

$$\exp(-2\pi w|\xi|^2) = \prod_{i=1}^{r_2} \exp(-2\pi w_{r_1+i}|\xi^{(r_1+i)}|^2), H_l(\sqrt{4\pi y}\xi) =$$

$$\prod_{i=1}^{r_1} H_{l_i}(\sqrt{4\pi \Im(z_i)}\xi^{(i)}) \text{ and } H_n(x) = (-1)^n \exp(x^2/2) \frac{d^n}{dx^n} \exp(-x^2/2).$$

Moreover, for $l \in \mathbb{Z}^{r_1}, u = (u_{r_1+1}, \cdots, u_{r_1+r_2}) \in \mathbb{C}^{r_2}$ and $(c, d) \in F \times F$, we also define $\vartheta(\mathfrak{z}, u; c, d)$ by

(2-8)
$$\vartheta_l(\mathfrak{z}, u; c, d) = y^{-l/2} \sum_{a \in \mathfrak{o}} H_l(\sqrt{4\pi y} a) \times e_s(a^2 z/2) e_c((1/2)(-2au + cu^2)d + (1/2)(cu - a)^2 z) \exp(-2\pi w|cu - a|^2),$$

where

$$
\begin{aligned}
&e_c((1/2)(-2au+cu^2)d+(1/2)(cu-a)^2z))\\
&\quad=\prod_{i=1}^{r_2} e[2\Re((1/2)(-2a^{(r_1+i)}u_{r_1+i}+c^{(r_1+i)}u_{r_1+i}^2)d^{(r_1+i)}\\
&\quad+(1/2)(c^{(r_1+i)}u_{r_1+i}-a^{(r_1+i)})^2 z_{r_1+i})]
\end{aligned}
$$

and

$$
\exp(-2\pi w|cu-a|^2)=\prod_{i=1}^{r_2}\exp(-2\pi w_{r_1+i}|c^{(r_1+i)}u_{r_1+i}-a^{(r_1+i)}|^2).
$$

Applying Poisson summation formula, $\Theta(\mathfrak{z},\mathfrak{w};\eta_\lambda)$ may split into a Poincarè series-type sum, which is an essentially role for our later computations.

Proposition 2.1. *Suppose that η satisfies (2-3). Then*

(2-9)

$$
\begin{aligned}
&(\sqrt{\pi}r)^m y^{(m-u_{r_1})/2}\overline{\Theta(\mathfrak{z},\mathfrak{w};\eta_\lambda)}\\
&\quad=(-1)^{\{m\}}\sum_{0\leq n\leq m}\binom{m}{n} i^{\{m-n\}}\sqrt{2\pi(-1)}^{n}(\sqrt{y/2}/\tilde{r})^{-n-u_{r_1}}\sqrt{|d_F|}\\
&\quad\times\sum_{(c,d)\in T_\lambda(\mathfrak{r}^{-1})} y^{(m-n)/2}\vartheta_{m-n}(\mathfrak{z},u;c,d)N(t_\lambda\mathfrak{c})\varphi^*(t_\lambda/2\mathfrak{r})\varphi_{\mathfrak{r}\mathfrak{c}}(d/2)2^{-n-d-r_2}\\
&\quad\times(c\bar{z}+d)^n(v^2/w)e_a(\sqrt{-1}\tilde{r}^2|cz+d|^2/4y)\exp(-(v^2/w)\pi(|cz+d|^2+|c|^2w^2)),
\end{aligned}
$$

where $T_\lambda(\mathfrak{r}^{-1})=\{(c,d)\in 2^{-1}t_\lambda\mathfrak{c}\mathfrak{d}\times t_\lambda\mathfrak{r}^{-1}\},\mathfrak{w}=(r_1\sqrt{-1},\cdots,r_{r_1}\sqrt{-1},u_{r_1+1}+jv_{r_1+1},\cdots,u_{r_1+r_2}+jv_{r_1+r_2}),\tilde{r}=(r_1,\cdots,r_{r_1})\in(\mathbb{R}^+)^{r_1}$ *and* $\mathfrak{z}=(z_1,\cdots,z_{r_1},z_{r_1+1}+jw_{r_1+1},\cdots,z_{r_1+r_2}+jw_{r_1+r_{r_2}})$.

By Shimura [9, Prop. 1.3], we may derive the following transformation formula.

Proposition 2.2.

(2-10) $\quad\tilde{J}_l(\beta\gamma,\beta^{-1}(\mathfrak{z}))\vartheta_l(\beta^{-1}(\mathfrak{z}),u,l(\beta\gamma))=\tilde{\vartheta}_l(\beta\gamma\beta^{-1}(\mathfrak{z}),u)$ *for every* $\gamma\in\Gamma_{\mathfrak{r}\mathfrak{c}}$,

where $\tilde{J}_l(\gamma,\mathfrak{z})=h(\gamma,\mathfrak{z})j(\gamma,\mathfrak{z})^l$ *and* $j(\gamma,\mathfrak{z})^l=\prod_{i=1}^{\gamma_1}(c_{\gamma_i}z_i+d_{\gamma_i})^{l_i}$*for a* $l\in\mathbb{Z}^{\gamma_1}$ *and* $\gamma\in G\cap pr^{-1}(P_{\mathbb{A}}C'')$.

Using Proposition 2.1 and 2.2, we conclude that

Theorem 2.3. *Let f be an element of $\mathcal{S}_{m+(1/2)u_{r_1},\omega}(\mathfrak{b},\mathfrak{b}';\psi)$. Suppose that $\tau \in F^\times, \tau \gg 0, \tau\mathfrak{b} = \mathfrak{q}^2\mathfrak{r}$ with a fractional ideal $\mathfrak{q}$ and a square free integral ideal $\mathfrak{r}$ and $m > 0$. Then*

(2-11)

$$g_{\tau,\lambda}(\mathfrak{w}) = N(t_\lambda/\mathfrak{r})\sum_{\mathfrak{m}}\sum_{l\in t_\lambda\mathfrak{r}^{-1}\mathfrak{m}} N(\mathfrak{m})l^{m-1}|l|^{-1}\varphi_a(l)\varphi^*(l\mathfrak{r}/t_\lambda\mathfrak{m})\mu_f(\tau,(\mathfrak{r}\mathfrak{q})^{-1}\mathfrak{m})e_s(lz)vK_{2\nu}(4\pi|l|v)e_c(lu),$$

where $\mathfrak{m}$ runs over all integral ideals and $\mathfrak{w} = (z_1,\cdots,z_{r_1},\mathfrak{z}_{r_1+1},\cdots,\mathfrak{z}_{r_1+r_2}), z = (z_1,\cdots,z_{r_1}), \mathfrak{z}_{r_1+i} = u_{r_1+i} + jv_{r_1+i}(1 \leqq i \leqq r_2), u = (u_{r_1+1},\cdots,u_{r_1+r_2}), v = (v_{r_1+1},\cdots,v_{r_1+r_2}), |l| = \prod_{i=1}^{r_2}|l^{(r_1+i)}|$ and $l^{m-1} = \prod_{i=1}^{r_1}(l^{(i)})^{m_i-1}$.

By the same method as in [8, p. 536], we may deduce the following.

Theorem 2.4. *Let f be an element of $\mathcal{S}_{m+(1/2)u_{r_1},\omega}(\mathfrak{b},\mathfrak{b}';\psi)$: let $0 \ll \tau \in F^\times$ be an element such that $\tau\mathfrak{b} = \mathfrak{q}^2\mathfrak{r}$ with a fractional ideal $\mathfrak{q}$ and a square free integral ideal $\mathfrak{r}$. Suppose that f is a common eigenform of $\mathbb{T}_v$ for each $v \in h$,i.e.,*

$$f|\mathbb{T}_v = \chi(v)N_v^{-1}f \quad \text{for every } v \in h.$$

Then there exists the normalized eigenform $\mathfrak{g}$ belonging to $\mathcal{S}_{2m,4\omega+3}(2^{-1}\mathfrak{c},\psi^2)$ attached to χ such that

$$\mu_f(\tau,\mathfrak{q}^{-1})\mathfrak{g} = (g_{\tau,1},\cdots,g_{\tau,\kappa}). \tag{2-12}$$

§3 Key lemmas of theta integrals and Eisenstein series. In this section, we show that a Hilbert modular form of half integral weight is expressed as an inner product of a theta function and the modular form attached to its image of the Shimura correspondence. For an integral ideal $\mathfrak{a}$ we define two elements $\zeta^{\mathfrak{a}}$ and $\zeta_{\mathfrak{a}}$ of $\mathcal{S}(V_h)$ by

$$\zeta^{\mathfrak{a}}(x) = \begin{cases} \overline{\varphi}_a(b_x)\overline{\varphi}^*((b_x\mathfrak{a}\mathfrak{e})) & \text{if } x \in \mathfrak{o}[\mathfrak{a}\mathfrak{e}^{-1},\mathfrak{e}], \\ 0 & \text{otherwise} \end{cases} \tag{3-1}$$

and

$$\zeta_{\mathfrak{a}}(x) = \begin{cases} \overline{\varphi}_a(b_x)\overline{\varphi}^*((b_x\mathfrak{a}^{-1}\mathfrak{e})) & \text{if } x \in \mathfrak{o}[\mathfrak{a}\mathfrak{e}^{-1},\mathfrak{e}] \text{ and } (b_x\mathfrak{a}^{-1}\mathfrak{e},\mathfrak{r}\mathfrak{c}) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Here we consider the following assumptions.

$$\psi_a(x) = (\mathrm{sgn}(x_s))^m|x_s|^{i\lambda}|x_c|^{2i\mu} \ (x \in F_a^\times) \text{ and } \mathfrak{r} \text{ divides } \mathfrak{h}, \tag{3-2}$$

where $(\mathrm{sgn}(x_s))^m = \prod_{i=1}^{r_1} \mathrm{sign}(x_i)^{m_i}$, $|x_s|^{i\lambda} = \prod_{i=1}^{r_1} |x_i|^{\sqrt{-1}\lambda_i} (x_s = (x_1, \cdots, x_{r_1}) \in F_s^{\times})$, $|x_c|^{2i\mu} = \prod_{i=1}^{r_2} |x_{r_1+i}|^{2\sqrt{-1}\mu_{r_1+i}} (x_c = (x_{r_1+1}, \cdots, x_{r_1+r_2}) \in F_c)$, $(\lambda_1, \cdots, \lambda_{r_1}, \mu_{r_1+1}, \cdots, \mu_{r_1+r_2}) \in \mathbb{R}^{r_1+r_2}$ and $\sum_{i=1}^{r_1} \lambda_i + \sum_{i=1}^{r_2} \mu_{r_1+i} = 0$.

If v is a common prime factor of 2 and $\mathfrak{r}$, then φ_v satisfies either

(3-3) $\quad (i) \quad (\mathfrak{r}\mathfrak{c})_v = \mathfrak{h}_v = 4\mathfrak{r}_v$ and $\varphi_v(1+4x) = \varphi_v(1+4x^2)$ for all $x \in \mathfrak{o}_v$: or

$(ii) \quad (\mathfrak{r}\mathfrak{c})_v \neq \mathfrak{h}_v \subset 4\mathfrak{r}_v$.

(3-4) If $f' \in \mathcal{S}_{m+(1/2)u_{r_1},\omega}(\mathfrak{b}, \mathfrak{b}'; \psi)$ and $f'|\mathbb{T}_v = N_v^{-1}\chi(v)f'$ for each $v \nmid \mathfrak{h}^{-1}\mathfrak{r}^2\mathfrak{c}$,

then f' is a constant times f.
(3-5)
If $0 \neq f' \in \mathcal{S}_{m+(1/2)u_{r_1},\omega}(\mathfrak{b}, \mathfrak{b}''; \psi)$ with a divisor $\mathfrak{b}''$ of $\mathfrak{b}'$ and $f'|\mathbb{T}_v = N_v^{-1}\chi(v)f'$

for every $v \nmid \mathfrak{h}^{-1}\mathfrak{r}^2\mathfrak{c}$, then $\mathfrak{b}'' = \mathfrak{b}'$ and f' is a constant times f. Furthermore, we consider the condition.

(3-6) $\quad 4\mathfrak{r}\mathfrak{b} \supset \mathfrak{h} \cap 4\mathfrak{o}$; $\mathfrak{h}^{-1}\mathfrak{r}\mathfrak{c}$ is prime to $\mathfrak{r}$; $\mathfrak{h}_v = \mathfrak{c}_v$ or $\mathfrak{c}_v \neq 4\mathfrak{o}_v$ if $v|2$ and $v \nmid \mathfrak{r}$.

By the same method as that of Shimura [7] and [8, Prop. 5.8] , we may derive the following.

Proposition 3.1. *Let $\mathfrak{a}$ be an integral ideal such that $\mathfrak{a}\mathfrak{h} \supset \mathfrak{r}\mathfrak{c}$ and $(\mathfrak{a}\mathfrak{h})_v = (\mathfrak{r}\mathfrak{c})_v$ for each $v|\mathfrak{r}$, and let $\mathfrak{g} = (g_1, \cdots, g_\kappa)$ be the element in Theorem 2.4. Suppose that the conditions (3-2), (3-3) and (3-4) are satisfied. Put*

(3-7)
$$l(\mathfrak{z}) = \sum_{\lambda=1}^{\kappa} \langle \Theta(\mathfrak{z}, \mathfrak{w}; \zeta_\lambda^{\mathfrak{a}}), g_\lambda(\mathfrak{w}) \rangle.$$

Then l coincides with $M_{\mathfrak{a}}h$ with a constant $M_{\mathfrak{a}}$ which is 0 if $4\mathfrak{r}\mathfrak{b} \supset \mathfrak{a}\mathfrak{h} \cap 4\mathfrak{o} \neq \mathfrak{r}\mathfrak{c}$ and (3-5) is assumed.

Using Proposition.3.1, we may confirm the following proposition.

Proposition 3.2. *Let f, h and $\mathfrak{g} = (g_1, \cdots, g_\kappa)$ be as above; let η and $\zeta_{\mathfrak{a}}$ be as in (3-1). Then, under (3-2), (3-3) and (3-4), we have*

(3-8)
$$Ah(\mathfrak{z}) = \sum_{\lambda=1}^{\kappa} \langle \Theta(\mathfrak{z}, \mathfrak{w}; \eta_\lambda), g_\lambda(\mathfrak{w}) \rangle$$

with

$$A = \overline{C\mu_f(\tau, \mathfrak{q}^{-1})}N(\mathfrak{q}\mathfrak{r})(\langle \mathfrak{g}, \mathfrak{g} \rangle / \langle f, f \rangle) vol(\Gamma[2\mathfrak{d}^{-1}, 2^{-1}\mathfrak{r}\mathfrak{c}\mathfrak{d}])^{-1} |\tau_c|^{-3} \tau_s^{-(m+(1/2)u_{r_1})},$$

where $\tau_s^{-(m+(1/2)u_{r_1})} = \prod_{i=1}^{r_1}(\tau^{(i)})^{-(m_i+1/2)}$ *and* $|\tau_c|^{-3} = \prod_{i=1}^{r_2}|\tau^{(r_1+i)}|^{-3}$.

Moreover, if in addition, (3-5) and (3-6) are assumed, then

(3-9) $$KAh(\mathfrak{z}) = \sum_{\lambda=1}^{\kappa}\langle\Theta(\mathfrak{z},\mathfrak{w};\zeta_{o\lambda}), g_\lambda(\mathfrak{w})\rangle$$

$$\textit{with} \quad K = \varphi_a(-1)\gamma(\varphi)\mu(\mathfrak{h}^{-1}\mathfrak{rc})\varphi^*(\mathfrak{h}^{-1}\mathfrak{rc})N(\mathfrak{rc})^{-1}.$$

Here we introduce Eisenstein series. Let ω be a Hecke character of F of conductor $\mathfrak{f}$ such that

(3-10) $$\omega_s(x) = (\mathrm{sgn}(x))^n|x|^{\sqrt{-1}\lambda} \quad (x \in F_s^\times) \text{ and } \omega_c(x) = |x|^{2\sqrt{-1}\mu} \quad (x \in F_c^\times),$$

where $n = (n_1, \cdots, n_{r_1}) \in \mathbb{Z}^{r_1}$, $\lambda = (\lambda_1, \cdots, \lambda_{r_1}) \in \mathbb{R}^{r_1}$ and $\mu = (\mu_{r_1+1}, \cdots, \mu_{r_1+r_2}) \in \mathbb{R}^{r_1}$. Given a function f on D and α in G, we put $f||_n\alpha(\mathfrak{z}) = (c_\alpha z + d_\alpha)^{-n}f(\alpha(\mathfrak{z}))(\mathfrak{z} \in D)$. We put

$$E(\mathfrak{z}, s : n, \omega, \Gamma^*) = \sum_{\alpha\in R}\omega_a(d_\alpha)\omega^*(d_\alpha\mathfrak{a}_\alpha^{-1})N(\mathfrak{a}_\alpha)^{2s}y^{su_{r_1}+(i\lambda-n)/2}w^{2su_{r_2}+i\mu}||_n\alpha,$$

$$E_\beta(\mathfrak{z}, s : n, \omega, \Gamma^*) = N(\mathfrak{a}_\beta)^{2s}\sum_{\alpha\in R_\beta}\omega_a(d_\alpha)\omega^*(d_\alpha\mathfrak{a}_\alpha^{-1})y^{su_{r_1}+(i\lambda-n)/2}w^{2su_{r_2}+i\mu}||_n\alpha,$$

where $R = P\backslash G \cap P_{\mathbb{A}}D^*, R_\beta = (P \cap \beta\Gamma^*\beta^{-1}\backslash\beta\Gamma^*$ and $\Gamma^* = G \cap D^*$ with a open subgroup D^* of $D[\mathfrak{x}^{-1}, \mathfrak{x}\mathfrak{l}]$. Moreover, for a fractional ideal $\mathfrak{x}$ and an integral ideal $\mathfrak{l}$, we put

(3-11)

$$C(\mathfrak{z}, s : n, \omega, \Gamma_0) = L_\mathfrak{l}(2s, \omega)E(\mathfrak{z}, s : n, \omega, \Gamma_0) \text{ and } L_\mathfrak{l}(s, \omega) = \sum_{\mathfrak{m}}\omega^*(\mathfrak{m})N(\mathfrak{m})^{-s},$$

where $\Gamma_0 = \Gamma[\mathfrak{x}^{-1}, \mathfrak{x}\mathfrak{l}]$ and $\mathfrak{m}$ runs over all integral ideals prime to $\mathfrak{l}$.

§4 The expression of $\mu_f(\tau, \mathfrak{q}^{-1})D(s, \chi)$ by Rankin's convolution, the image of the product of Eisenstein series theta functions under the Shimura correspondence and the final calculations. Here we express $\mu_f(\tau, \mathfrak{q}^{-1})D(s, \chi)$ as a Rankin's convolution of a theta series and h. We define a theta series $\vartheta(\mathfrak{z})$ by

(4-1) $$\vartheta(\mathfrak{z}) = \sum_{b\in\mathfrak{o}} e_s(b^2z/2)e_c(b^2z/2)\exp(-2\pi w|b|^2),$$

where $\mathfrak{z} = (z_1, \cdots, z_{r_1}, \mathfrak{z}_{r_1+1}, \cdots, \mathfrak{z}_{r_1+r_2})$ $(\mathfrak{z}_{r_1+i} = z_{r_1+i} + jw_{r_1+i})$, $e_s(b^2z/2) = \prod_{i=1}^{r_1}e[(b^{(i)})^2z_i/2], e_c(b^2z/2) = \prod_{i=1}^{r_2}e[\Re((b^{(r_1+i)})^2z_{r_1+i})]$ and $\exp(-2\pi w|b|^2) =$

$\prod_{i=1}^{r_2}\exp(-2\pi w_{r_1+i}|b^{(r_1+i)}|^2)$. From some computations, we may deduce that

(4-2)
$$\int_\Phi h(\mathfrak{z})\overline{\vartheta(\mathfrak{z})C(\mathfrak{z},\overline{s}+1/2:m,\overline{\varphi},\Gamma)}y^{m+(1/2)u_{r_1}}w^2d\mathfrak{z}$$
$$=(2\pi)^{-r_1s-(m+i\lambda)/2}(2\pi)^{-2sr_2-i\mu-r_2}2^{1+d-(3/2)r_2-2sr_2-i\mu}$$
$$\times\pi^{r_2/2}\sqrt{|d_F|}^{-1}\Gamma'(s+(i\lambda+m)/2)D(2s,\chi)\mu_f(\tau,\mathfrak{q}^{-1})$$
$$\times N(\mathfrak{r})^{-2s}\frac{\Gamma'(2s+i\mu-\nu+1/2)\Gamma'(2s+i\mu+\nu+1/2)}{\Gamma'(2s+i\mu+1)},$$

where $\Gamma=\Gamma_{\mathfrak{r}\mathfrak{c}}$ and $\Phi=\Gamma_{\mathfrak{r}\mathfrak{c}}\backslash D$. On the other hand, by (3-9), we confirm

(4-3)
$$KA\int_\Phi h(\mathfrak{z})\overline{\vartheta(\mathfrak{z})C(\mathfrak{z},\overline{s}+1/2;m,\overline{\varphi},\Gamma)}y^{m+(1/2)u_{r_1}}w^2d\mathfrak{z}$$
$$=\int_\Phi\sum_{\lambda=1}^{\kappa}\langle\Theta(\mathfrak{z},\mathfrak{w}:\zeta_{o^\lambda}),g_\lambda(\mathfrak{w})\rangle\vartheta(\mathfrak{z})\overline{C(\mathfrak{z},\overline{s}+1/2;m,\overline{\varphi},\Gamma)}y^{m+(1/2)u_{r_1}}w^2d\mathfrak{z}$$
$$=\sum_{\lambda=1}^{\kappa}\langle M'_\lambda(\mathfrak{w},\overline{s}),g_\lambda(\mathfrak{w})\rangle$$

with $M'_\lambda(\mathfrak{w},s)=\int_\Phi\vartheta(\mathfrak{z})\Theta(\mathfrak{z},\mathfrak{w};\zeta_{o^\lambda})C(\mathfrak{z},s+1/2,m,\overline{\varphi},\Gamma)y^{m+(1/2)u_{r_1}}w^2d\mathfrak{z}$. We may derive that

(4-4)
$$\sum_{\lambda=1}^{\kappa}\langle M'_\lambda(\mathfrak{w},\overline{s}),g_\lambda(\mathfrak{w})\rangle=(\pi/2)^{-su_{r_1}-(u_{r_1}+m+i\lambda)/2}\pi^{-2su_{r_2}-i\mu-u_{r_2}}$$
$$\times 2^{1-r_2+d}\sqrt{|d_F|}^{-1}\Gamma'(s+(1+m+i\lambda)/2)\Gamma'(2s+i\mu+1)$$
$$\times L_{\mathfrak{r}\mathfrak{c}}(2s+1,\varphi)\sum_{\lambda=1}^{\kappa}\langle\sum_{\beta\in B}\varphi^*(\mathfrak{a}_\beta)N(\mathfrak{a}_\beta)^{2\overline{s}+1}S'_{\beta,\lambda}(\mathfrak{w},\overline{s}),g_\lambda(\mathfrak{w})\rangle,$$

where

$$S'_{\beta,\lambda}(\mathfrak{w},s)=\sum_{(\xi,\alpha)\in X/o\times}\zeta_{o\lambda}(p\xi)\mu_\beta(\alpha)[\xi,\mathfrak{w}]^{-m}$$
$$\times|[\xi,\mathfrak{w}]/\eta(\mathfrak{w})|^{-2((s+1/2)u_{r_1}+(m+i\lambda)/2)}|[\xi+\alpha I_2,\mathfrak{w}]/\eta(\mathfrak{w})|^{-2((2s+1)u_{r_2}-i\mu)},$$

$G\cap P_{\mathbb{A}}D_{\mathfrak{r}\mathfrak{c}}=\bigsqcup_{\beta\in B}P_+\beta\Gamma_{\mathfrak{r}\mathfrak{c}}$, $P_+=\{\alpha\in P|d_\alpha\gg 0\}$, $X=\{(\xi,\alpha)\in V\times F|-\det(\xi)=\alpha^2\}$ and μ_β is the characteristic function of $\mathfrak{a}_\beta$. Here we take β from $\mathrm{diag}[p,p^{-1}]U$ with $p\in F_h^\times$ such that $(p_v)_{v|\mathfrak{h}}=1$ with any small open subgroup U of $G_{\mathbb{A}}$. By the same method as that of [8, p. 545-549], we may verify that

(4-5)
$$(-1)^{\{m\}}S'_{\beta,\lambda}(\mathfrak{w},s)=\sum_{q\in Q}(q\mathfrak{w}+1)^{2m}T'_{\beta,\lambda}(\tau_q(\mathfrak{w}),s)\quad\text{with}$$

$$\sum_{\beta\in B}\varphi^*(\mathfrak{a}_\beta)N(\mathfrak{a}_\beta)^{2s}T'_{\beta,\lambda}(\mathfrak{w},s-1/2)$$
$$=N(\mathfrak{e}_\lambda)^{2s}C(\mathfrak{w},s:m,\overline{\varphi},\Gamma_\lambda)E(\mathfrak{w},s:m,\overline{\varphi},\Gamma_\lambda),$$

where $Q=\mathfrak{e}_\lambda/2\mathfrak{e}_\lambda, \tau_q=\begin{pmatrix}1&0\\q&1\end{pmatrix})(q\in Q)$ and $\Gamma_\lambda=\Gamma[\mathfrak{r}\mathfrak{c}\mathfrak{e}_\lambda^{-1},2\mathfrak{e}_\lambda]$. By (4-4) and (4-5), we deduce that

$$\sum_{\lambda=1}^{\kappa}\langle\sum_{\beta\in B}\varphi^*(\mathfrak{a}_\beta)N(\mathfrak{a}_\beta)^{2\overline{s}+1}S'_{\beta,\lambda}(\mathfrak{w},\overline{s}),g_\lambda(\mathfrak{w})\rangle \tag{4-6}$$
$$=\sum_{\lambda=1}^{\kappa}(-1)^{\{m\}}\sharp Q\langle N(\mathfrak{e}_\lambda)^{2\overline{s}+1}C(\mathfrak{w},\overline{s}+1/2:m,\overline{\varphi},\Gamma_\lambda)$$
$$\times E(\mathfrak{w},\overline{s}+1/2:m,\overline{\varphi},\Gamma_\lambda),g_\lambda(\mathfrak{w})\rangle.$$

Using an explicit calculation of Fourier coefficients of Eisenstein series and a Rankin convolution, we may find that (4-6) is equal to

$$M(s)N(\mathfrak{r})^{-2s}L_{\mathfrak{c}}(2s+1,\varphi)^{-1}D(2s,\chi)D(0,\chi,\overline{\varphi})\sum_{\mathfrak{t}\supset\mathfrak{h}^{-1}\mathfrak{r}\mathfrak{c}}\mu(\mathfrak{t})\varphi^*(\mathfrak{t})\chi(\mathfrak{t}^{-1}\mathfrak{h}^{-1}\mathfrak{r}\mathfrak{c}), \tag{4-7}$$

where $M(s)$ is an explicitly defined factor which is a product of an arithmetical quantity of F, exponential functions and gamma functions of s. Consequently, by (3-8),(3-9), (4-2), (4-3), (4-4) and (4-7), we conclude the following theorem.

Theorem 4.1. *Let f be an element satisfying the conditions in Theorem 2.4, and let τ be an element of $F^\times$ such that $\tau\gg 0$, $\tau\mathfrak{b}=\mathfrak{q}^2\mathfrak{r}$ with a fractional ideal $\mathfrak{q}$ and a square free integral ideal $\mathfrak{r}$. Suppose that the assumptions (3-2)∼ (3-6) are satisfied. Then*

$$|\mu_f(\tau,\mathfrak{q}^{-1})|^2\varphi^*(\mathfrak{h}^{-1}\mathfrak{r}\mathfrak{c})\mu(\mathfrak{h}^{-1}\mathfrak{r}\mathfrak{c})=RN(\mathfrak{r}\mathfrak{q})^{-1}D(0,\chi,\overline{\varphi})\langle f,f\rangle/\langle\mathfrak{g},\mathfrak{g}\rangle \tag{4-8}$$

with $$R=\pi^{-\{m\}}2^{-1+(r_1/2)+2r_2-\{m\}}\tau_s^{m+(1/2)u_{r_1}}$$
$$\times|\tau_c|^3\Gamma'(m)\Gamma'(\nu+1/2)\Gamma'(-\nu+1/2)[\mathfrak{o}_+^\times:(\mathfrak{o}^\times)^2]h_F$$
$$\times\frac{vol([\Gamma[2\mathfrak{d}^{-1},2^{-1}\mathfrak{r}\mathfrak{c}\mathfrak{d}]\backslash D)}{vol([\Gamma[2\mathfrak{r}\mathfrak{d}^{-1},\mathfrak{c}\mathfrak{d}]\backslash D)}\sum_{\mathfrak{t}\supset\mathfrak{h}^{-1}\mathfrak{r}\mathfrak{c}}\mu(\mathfrak{t})\varphi^*(\mathfrak{t})\chi(\mathfrak{t}^{-1}\mathfrak{h}^{-1}\mathfrak{r}\mathfrak{c}).$$

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